A BIASED ESTIMATE OF THE PROCESS AVERAGE

BY

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1. Summary.

The purpose of this report is to investigate existing estimates of the process average and to propose more sensitive criteria for tightened and reduced[1/] inspection under the double sampling plans of MIL-STD-105A, Sampling Procedures and Tables for Inspection by Attributes.

Under the present MIL-STD-105A procedure for double sampling an estimate of the process average is computed solely on the basis of the first samples from preceding lots. This unbiased estimate is the ratio of the total number of defective items found in the first samples to the total number of items inspected in all of the first samples. If this average falls above the upper limit given for the specified AQL, then tightened inspection is begun. It seems reasonable that "better" criteria can be obtained by using an estimate based on both samples, since the combined sample contains three times the number of items in the first sample.[2/] The term "better" is used in the sense that when quality deteriorates, the probability of going on tightened inspection should be higher than that under the present system.

[1/] In its present form, MIL-STD-105A does not provide for reduced inspection under a double sampling plan. However, the present system can be used for finding lower limits for the estimated process average. If the estimate is below this lower limit, reduced inspection can then be instituted. Similarly, criteria for reduced inspection can be obtained by using the procedures presented in this report.

[2/] The results of this paper are based on uncurtailed sampling plans only. That is when the procedure calls for a second sample, the total second sample will be inspected and not just part of it.

A more natural estimate of the process average to use when double sampling is employed is the ratio of the total number of defective items found in both samples from preceding lots to the total number of items inspected. This estimate can be shown to be "biased". That is, if the estimation procedure is repeated over and over again, the average value of the above estimate will be different from the true process average. However, it will be shown that this biased estimate based on the combined samples will be closer, on the average, to the true process average than the unbiased estimate based solely on the first samples.

This report includes a table of the upper limits of the process average for double sampling plans -- Table 1. These limits are based on the biased estimate of the process average described above. Evidence is presented to show that, at least for the range of AQLs included in this report, these limits are "better", in the above sense, than those found in MIL-STD-105A.

Table 2 gives the bias of the above estimate based on both samples, for each AQL and sample size code letter. The bias is defined as the difference between the true process average and the average value of this estimate and is given here for the process average equal to the AQL. It is interesting to note that the bias is always positive. That is, the biased estimate underestimates the true process average, within the range of this table.

In Table 3, a study is made of how "close", on the average, the biased and the unbiased estimates are to the true process average. That is, it compares the respective mean square deviations from the true process average, called the mean square error. It is evident that for each plan considered, the biased estimate is, on the average, "closer" in the above sense.

2. <u>Procedure For Tightened Inspection When Double Sampling Is Used</u>.

1. After a sufficient number of lots have been inspected according to the procedures in MIL-STD-105A for double sampling, estimate the process average by computing the ratio of the total number of defectives found in all the lots, D, to the total number of items inspected, N. Call this estimate p_{bk}, where k is the number of lots used to compute p_{bk}. Convert p_{bk} to percent by multiplying it by 100.

2. Enter Table 1 with the AQL, sample size code letter,[1/] and the number of lots, k, used to compute p_{bk} to find the appropriate upper limit for the process average. Table 1 gives limits for k = 5, 10, and 15.

3. If p_{bk}, in percent, is above this limit, change to tightened inspection.

For example, consider the following results of inspection using the sampling plan of MIL-STD-105A with code letter F and AQL = 6.5%.

$$n_1 = 10\ ,\ a_1 = 1\ ,\ r_1 = 4\ ,\ n_2 = 20\ ,\ a_2 = 3\ ,\ r_2 = 4$$

Lot	Number of defectives 1st sample	2nd sample	Total defectives both samples	Decision
1	2	1	3	Accept
2	0	-	0	Accept
3	2	0	2	Accept
4	3	1	4	Reject
5	1	-	1	Accept
6	2	0	2	Accept
7	0	-	0	Accept
8	1	-	1	Accept
9	4	-	4	Reject
10	2	0	2	Accept

[1/] If the sample size code letter is not the same for all samples used, the entry in Table 1 is determined by the code letter of the smallest sample used.

$D = 19.$ $N = 10(10)+5(20) = 200.$

$p_{b,10} = \frac{D}{N} = .095$ or 9.5%.

Upper limit from Table 1 for $k = 10$ is 10.604%.

Since p_{bk} is less than the upper limit, normal inspection should be continued.

3. Notation Used in This Report.

p	true fraction defective[1/]
$q = 1-p$	true fraction non-defective
p_1	unbiased[2/] estimate of p based on the first sample from one lot
p_b	biased estimate of p based on both samples from one lot
p_{bk}	biased estimate of p based on both samples from each of k lots
p_u	unbiased estimate of p based on both samples from one lot
a_1	acceptance number for first sample
a_2	acceptance number for second sample
r_1	rejection number for first sample
r_2	rejection number for second sample
d_1	number of defectives in first sample
d	total number of defectives in both samples from one lot
D	total number of defectives in both samples from each of k lots
n_1	size of first sample
n_2	size of second sample
n	total number of items inspected in one lot

1/ This parameter is often called p′ in Quality Control work. It should be emphasized that p_1, p_b, and p_u are not parameters but statistics which estimate p.

2/ An unbiased estimate is one such that, if the estimation procedure is repeated over and over again, the average value of the estimate will be equal to the true fraction defective.

N total number of items inspected in k lots

AQL acceptable quality level, as defined in MIL-STD-105A.

4. Estimates of the Fraction Defective.

Consider three estimates of the fraction defective p based on an uncurtailed double sampling plan for inspection by attributes:

1. $$p_1 = \frac{d_1}{n_1}$$

2. $$p_u = \frac{d_1}{n_1}$$ when a decision is made on one sample only, $r_1 \leq d_1 \leq a_1$

$$p_u = \frac{\sum_{y_o = a_1+1}^{r_1-1} \binom{n_1-1}{y_c-1}\binom{n_2}{d-y_o}}{\sum_{y_o = a_1+1}^{r_1-1} \binom{n_1}{y_o}\binom{n_2}{d-y_o}}$$ when two samples are taken, where

$$\binom{m}{s} = \frac{m!}{s!(m-s)!} \quad , \; a_1 < d < n_2 + r_1 .$$

3. $$p_b = \frac{d}{n}$$ $d = 0,1,\ldots,a,r_1,\ldots,n_1$ when $n = n_1$

$d = a+1,\ldots,n_2+r_1-1$ when $n = n_1+n_2$

$$p_{bk} = \frac{D}{N} = \frac{\sum_{i=1}^{k} d_i}{\sum_{i=1}^{k} n_i} .$$

The estimate p_1 is an unbiased one based on the first sample only. It is the one on which the estimate in MIL-STD-105A is based. p_u is also unbiased, but it takes into account the additional information provided by the second sample. Since the computation involved in finding p_u is relatively difficult, it is rarely used in practical applications. The tables in this report are based on the third estimate, p_b or p_{bk}. It is a biased estimate based on both samples. In the case where a decision can be made from the first sample only, $p_b = p_u = p_1$.

It will be shown in Section 6 that, for sufficiently small values of the fraction defective, p, the mean square error of p_b is less than or equal to that of either p_1 or p_u.[1/] From this it follows directly that the variance of p_b is less than or equal to that of either of the other estimates.[2/] Thus, within this range of p, p_b, although biased, is a better estimate of p than are p_1 and p_u, because any limits based on its variance will be narrower, and therefore more sensitive for detecting a process average significantly different from the AQL. In addition, having a smaller mean square error, p_b will be closer on the average to the true fraction defective p than p_1 or p_u.

5. Preparation and Use of the Tables.

Table 1 gives three sets of upper limits for the process average. The first set applies if the process average is estimated on the basis of five lots, the second for ten lots, and the third for fifteen lots. Limits are given for each AQL and each sample size code letter.

The upper limit is a "three-sigma" limit, and is given by the expression $E(p_{bk})+3\sigma_{p_{bk}}$ where $E(p_{bk})$ is the expected (average) value of p_{bk} and $\sigma_{p_{bk}}$ is the standard deviation[3/] of the estimate. Both $E(p_{bk})$ and $\sigma_{p_{bk}}$ are computed for p equal to the AQL.

1/ $E(p_b-p)^2 \leq E(p_1-p)^2$ and $E(p_b-p)^2 \leq E(p_u-p)^2$ for some range of small p.

2/ $E[p_b-E(p_b)]^2 \leq E[p_1-p]^2$ and $E[p_b-E(p_b)]^2 \leq E[p_u-p]^2$ for some range of small p.

3/ $\sigma_{p_{bk}} = \sqrt{E[p_{bk}-E(p_{bk})]^2} = \frac{\sigma_{p_{b1}}}{\sqrt{k}}$.

The limits are such that, if the process average is at the AQL, the probability of an estimate p_{bk} falling above its limit is small (about .00135). In this respect, they resemble those of MIL-STD-105A. However, since the standard deviation and the mean square error of p_{bk} are less than those of the old estimate p_1,[1/] (at least for AQL's within the range of the tables), these limits are stricter, and the probability of detecting a deviation of p from the AQL is therefore greater than it is when p_1 is used.

Table 2 gives, for each AQL and sample size code letter, the bias of the estimate p_b, i.e., the difference between the true fraction defective p and the expected value of p_b. This is independent of k and is expressed in the form $E(p-p_b)$ or $p-E(p_b)$.

Using Table 3, one can compare the mean square errors of the two estimates p_b and p_1. These are expressed by $E(p_b-p)^2$ and $E(p_1-p)^2$. It is seen that in every sampling plan considered in this report, p_b has the lower mean square error.

6. Comparison of the Mean Square Errors of p_b and p_1.

It now remains to prove the following theorem:

$$E(p_b-p)^2 \leq E(p_1-p)^2 \quad \text{for some range of small } p, \tag{6.1}$$

where $E(p_b-p)^2 = E(\frac{d}{n}-p)^2$ is the mean square error of $p_b = \frac{d}{n}$, the biased estimate of p based on two samples, and $E(p_1-p)^2 = \frac{pq}{n_1}$ is the mean square error (or variance) of $p_1 = \frac{d_1}{n_1}$, the unbiased estimate based on the first sample.

Theorem. $E(\frac{d}{n}-p)^2 \leq \frac{pq}{n_1}$ for some range of small p.

[1/] See Table 3.

<u>Proof</u>. Case I: $a_1 > 0$

$$E(\frac{d}{n} - p)^2 = \frac{pq}{n_1} - \left\{\frac{1}{n_1^2} - \frac{1}{(n_1+n_2)^2}\right\} \sum_{d=a_1+1}^{n_1-1} (d-n_1p)^2 \binom{n_1}{d} p^d q^{n_1-d}$$

$$+ \frac{n_2 pq}{(n_1+n_2)^2} \sum_{d=a_1+1}^{n_1-1} \binom{n_1}{d} p^d q^{n_1-d} \quad .$$

We wish to discover under what condition on p is

$$\left\{\frac{1}{n_1^2} - \frac{1}{(n_1+n_2)^2}\right\} \sum_{d=a_1+1}^{r_1-1} (d-n_1p)^2 \binom{n_1}{d} p^d q^{n_1-d} \geq \frac{n_2 pq}{(n_1+n_2)^2} \sum_{d=a_1+1}^{r_1-1} \binom{n_1}{d} p^d q^{n_1-d}$$

or

$$(6.2) \qquad \frac{n_2+2n_1}{n_1^2} \sum_{d=a_1+1}^{r_1-1} (d-n_1p)^2 \binom{n_1}{d} p^d q^{n_1-d} \geq pq \sum_{d=a_1+1}^{r_1-1} \binom{n_1}{d} p^d q^{n_1-d} \quad .$$

This inequality is always satisfied if $(d-n_1p)^2 \geq 1$ and if $\frac{n_2+2n_1}{n_1^2} \geq pq$.

Now, $(d-n_1p)^2 \geq 1$ implies that $-1 \geq d-n_1p \geq 1$ or $\frac{d+1}{n_1} \leq p \leq \frac{d-1}{n_1}$, $d = a_1+1$, $\ldots, r_1-1$. Hence, upon inserting the largest value for d on the left side of this inequality, and the smallest value for d on the right side, we obtain the condition

$$(6.31) \qquad \frac{r_1}{n_1} \leq p \leq \frac{a_1}{n_1} \quad .$$

The condition $pq \leq \frac{n_2+2n_1}{n_1^2}$ reduces to the quadratic inequality in p,

$$p^2 - p + \frac{n_2+2n_1}{n_1^2} \geq 0 \qquad 0 \leq p \leq 1 \quad .$$

Upon setting the left side of this inequality equal to zero, and solving for p, we find that the inequality is satisfied if either

$$(6.32)\qquad p \le \frac{n_1-\sqrt{n_1^2-4n_2-8n_1}}{2n_1} \qquad \text{or} \qquad p \ge \frac{n_1+\sqrt{n_1^2-4n_2-8n_1}}{2n_1}$$

and it is true for all p between zero and one if $n_1^2-4n_2-8n_1 \le 0$, that is, if $n_2 \ge \frac{n_1(n_1-8)}{4}$. Inequalities (6.31) and (6.32) are conditions on p which, if they are satisfied, are sufficient to guarantee that $E(\frac{d}{n}-p)^2 \le \frac{pq}{n_1}$.

A somewhat different condition on p can be found by substituting $\sum_{d=a_1+1}^{r_1-1}(d-n_1p)^2\binom{n_1}{d}p^dq^{n_1-d}$ in (6.2),

$$(a_1^2-2a_1n_1p)\sum_{j=1}^{b}\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}+2a_1\sum_{j=1}^{b}j\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}$$

$$+\sum_{j=1}^{b}(j-n_1p)^2\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}\quad,$$

where $b=r_1-1-a_1$. Inequality (6.2) becomes

$$(6.4)\qquad \left(\frac{n_2+2n_1}{n_1^2}\right)\left\{\left(a_1^2-2a_1n_1p-\frac{n_1^2pq}{n_2+2n_1}\right)\sum_{j=1}^{b}\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}\right.$$

$$\left.+\,2a_1\sum_{j=1}^{b}j\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}+\sum_{j=1}^{b}(j-n_1p)^2\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}\right\}$$

$$\ge 0\quad.$$

Since the second and third terms are positive for all p, this inequality always holds if

$$a_1^2-2a_1n_1p-\frac{n_1^2pq}{n_2+2n_1} \ge 0$$

or

$$n_1^2p^2-p\left\{n_1^2+2a_1n_1(n_2+2n_1)\right\}+a_1^2(n_2+2n_1) \ge 0\quad.$$

By setting the left hand side equal to zero, and solving for p, we find that (6.4) is always satisfied if

$$(6.41)\qquad p \le \frac{n_1+2a_1(n_2+2n_1)}{2n_1} - \frac{\sqrt{n_1^2(4a_1+1)^2+4a_1n_1(n_2+4a_1n_2-2a_1)+4a_1^2n_2(n_2-1)}}{2n_1}.$$

The relationships (6.32) and (6.41) will never yield negative values for p since the inequalities $p^2 - p + \frac{n_2+2n_1}{n_1^2} \ge 0$ and $n_1^2p^2-p\left\{n_1^2+2a_1n_1(n_2+2n_1)\right\}+a_1^2(n_2+2n_1) \ge 0$ are both satisfied for $p=0$.

Hence, for some interval about $p=0$, $E(\frac{d}{n}-p)^2 \le \frac{pq}{n_1}$.

Case II: $a_1 = 0$.

As in Case I, the inequality

$$\varphi(p) = \left\{a_1^2-2a_1n_1p-\frac{n_1^2(p-p^2)}{n_2+2n_1}\right\}\sum_{j=1}^{b}\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}$$

$$+2a_1\sum_{j=1}^{b} j\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j}+\sum_{j=1}^{b}(j-n_1p)^2\binom{n_1}{a_1+j}p^{a_1+j}q^{n_1-a_1-j} \ge 0$$

must be satisfied. When $a_1 = 0$, this becomes

$$\varphi(p|a_1=0) = -\frac{n_1^2(p-p^2)}{n_2+2n_1}\sum_{j=1}^{r_1-1}\binom{n_1}{j}p^jq^{n_1-j}+\sum_{j=1}^{r_1-1}(j-n_1p)^2\binom{n_1}{j}p^jq^{n_1-j} \ge 0.$$

The function $\varphi(p|a_1=0)$ is a continuous function. Taking the derivative of $\varphi(p|a_1=0)$ with respect to p, we find

$$\varphi'(p|a_1=0) = -\frac{n_1^2+pq}{n_2+2n_1}\sum_{j=1}^{r_1-1}\binom{n_1}{j}\frac{(j-pn_1)}{pq}\;p^jq^{n_1-j}$$

$$+\left(-\frac{n_1^2}{n_2+2n_1}+\frac{2n_1^2p}{n_2+2n_1}\right)\sum_{j=1}^{r_1-1}\binom{n_1}{j}p^jq^{n_1-j}+\sum_{j=1}^{r_1-1}(j-n_1p)^2\binom{n_1}{j}\frac{(j-n_1p)}{pq}\;p^jq^{n_1-j}$$

$$+\sum_{j=1}^{r_1-1}(-2n_1)(j-n_1p)\binom{n_1}{j}p^jq^{n_1-j} \qquad ,$$

and, setting $p=0$, $\varphi'(0|a_1=0)=\binom{n_1}{1}=n_1>0$.

Therefore, since $\varphi(p|a_1=0)$ is an increasing function of p in at least a small interval about $p=0$, and since $\varphi(0|a_1=0)=0$, the function must be positive for all p in some interval about $p=0$. For these values of p, $E(\frac{d}{n}-p)^2 \leq \frac{pq}{n_1}$.

A similar proof can be given for the theorem that $E(p_b-p)^2 \leq E(p_u-p)^2$ for some range of small p.

Table 1. Upper limits for the process average for double sampling when the process average is based upon the results of both samples in the preceding 5, 10, or 15 lots. When 5 lots have been used, read the black figures, for 10 lots read the red figures, and for 15 lots read the green figures.

Sample Size Code Letter	Acceptable Quality Level .015	.035	.065	.10	.15	.25	.40	.65	1.0	1.5	2.5	4.0	6.5	10.0
D														23.389
														19.041
														17.114
E													12.865	18.553
													10.266	15.427
													9.114	14.042
F												7.992	12.851	19.268
												6.336	10.604	16.252
												5.602	9.608	14.916
G											5.037	8.197	10.939	18.129
											3.980	6.732	9.199	15.549
											3.512	6.083	8.428	14.407
H										4.023	5.071	7.867	11.395	16.861
										3.223	4.167	6.609	9.804	14.751
										2.869	3.767	6.051	9.099	13.816
I									2.034	3.581	4.361	6.612	9.747	15.444
									1.602	2.904	3.644	5.666	8.553	13.726
									1.410	2.605	3.327	5.247	8.024	12.965
J								1.322	2.219	2.708	4.335	6.334	9.749	14.472
								1.038	1.806	2.252	3.692	5.518	8.670	13.077
								0.913	1.623	2.050	3.407	5.157	8.192	12.459
K							0.814	1.594	1.808	2.728	4.489	6.255	9.186	14.005
							0.637	1.289	1.503	2.309	3.870	5.521	8.285	12.772
							0.559	1.154	1.368	2.124	3.596	5.196	7.886	12.225
L						0.509	0.982	1.245	1.936	2.723	4.234	6.151	9.143	13.155
						0.396	0.791	1.028	1.627	2.329	3.695	5.478	8.305	12.143
						0.346	0.707	0.932	1.490	2.155	3.456	5.179	7.934	11.695
M					0.306	0.673	0.988	1.266	1.815	2.654	3.949	5.953	8.825	12.772
					0.238	0.539	0.808	1.063	1.552	2.302	3.502	5.365	8.109	11.910
					0.207	0.480	0.729	0.973	1.436	2.146	3.304	5.104	7.792	11.528
N				0.205	0.442	0.540	0.857	1.260	1.744	2.483	3.786	5.489	8.374	12.408
				0.159	0.351	0.441	0.712	1.069	1.509	2.181	3.392	5.017	7.779	11.660
				0.138	0.311	0.396	0.648	0.984	1.405	2.047	3.218	4.809	7.516	11.328
O			0.134	0.295	0.342	0.553	0.803	1.146	1.656	2.291	3.521	5.314	8.065	11.915
			0.103	0.234	0.277	0.457	0.679	0.990	1.454	2.046	3.205	4.910	7.573	11.313
			0.090	0.207	0.249	0.415	0.623	0.921	1.365	1.938	3.065	4.731	7.355	11.046
P		0.073	0.176	0.216	0.331	0.493	0.742	1.041	1.516	2.114	3.276	4.981	7.817	11.568
		0.056	0.141	0.176	0.274	0.417	0.639	0.919	1.358	1.924	3.033	4.675	7.416	11.086
		0.049	0.125	0.159	0.249	0.384	0.593	0.864	1.288	1.839	2.926	4.539	7.239	10.872
Q	0.032	0.092	0.124	0.192	0.296	0.424	0.616	0.944	1.350	1.951	3.081	4.697	7.413	11.152
	0.024	0.073	0.103	0.162	0.252	0.370	0.548	0.854	1.241	1.813	2.903	4.480	7.133	10.8[illegible]3
	0.021	0.065	0.093	0.148	0.232	0.346	0.518	0.814	1.192	1.752	2.825	4.383	7.009	10.649

Note: all figures in the table are read in percent.

Table 2

The Bias[1/] of the Biased Estimate of the Process Average Based on Both Samples of a Double Sampling Plan

Sample Size Code Letter	Acceptable Quality Level .015	.035	.065	.10	.15	.25	.40	.65	1.0	1.5	2.5	4.0	6.5	10.0
D														1.458
E													2.511	2.118
F												1.662	1.322	1.028
G											1.072	0.805	1.502	0.678
H										0.207	0.514	0.430	0.537	0.344
I									0.442	0.228	0.585	0.618	0.829	0.422
J								0.297	0.192	0.350	0.362	0.451	0.436	0.292
K							0.189	0.097	0.232	0.200	0.124	0.251	0.390	0.206
L						0.126	0.069	0.145	0.118	0.121	0.106	0.148	0.218	0.299
M					0.078	0.035	0.025	0.076	0.082	0.049	0.077	0.056	0.119	0.172
N				0.054	0.019	0.050	0.038	0.042	0.059	0.049	0.058	0.121	0.157	0.147
O			0.035	0.013	0.029	0.022	0.023	0.037	0.033	0.045	0.059	0.064	0.116	0.141
P		0.019	0.010	0.020	0.013	0.015	0.010	0.027	0.024	0.036	0.052	0.064	0.052	0.078
Q	0.009	0.006	0.015	0.012	0.005	0.011	0.016	0.014	0.023	0.020	0.026	0.045	0.044	0.039

[1/] The bias is computed for the process average equal to the AQL and is given by AQL-$E(p_{bk})$ where p_{bk} is the biased estimate being considered, and $E(p_{bk})$ is the average value of p_{bk}.

Note: All values in the table are read in percent.

Table 3. Comparison of the "Mean Square Errors from the True Process Average"[1/] between 1) the unbiased estimate of the true process average based on the first sample and 2) the biased estimate of the true process average based on both samples for double sampling plans. The top figure in each cell refers to the unbiased estimate, while the bottom figure refers to the biased estimate.

Sample Size Code Letter	Acceptable Quality Level													
	.015	.035	.065	.10	.15	.25	.40	.65	1.0	1.5	2.5	4.0	6.5	10.0
D														180.0
														124.6
E													86.8	128.6
													50.1	67.7
F												38.4	60.8	90.0
												20.5	34.5	59.9
G											16.2	25.6	40.5	60.0
											8.4	14.6	21.9	43.5
H										5.9	9.8	15.4	24.3	36.0
										4.2	5.6	10.4	16.7	29.0
I									2.8	4.2	7.0	11.0	17.4	25.7
									1.4	3.0	3.7	6.2	9.9	19.3
J								1.3	2.0	3.0	4.9	7.7	12.2	18.0
								0.6	1.1	1.5	2.8	4.5	7.7	12.7
K							0.5	0.9	1.3	2.0	3.2	5.1	8.1	12.0
							0.2	0.6	0.7	1.2	2.5	3.6	5.4	9.9
L						0.2	0.4	0.6	1.0	1.5	2.4	3.8	6.1	9.0
						0.1	0.2	0.3	0.6	1.0	1.9	3.0	4.6	6.7
M					0.100	0.166	0.266	0.431	0.660	0.985	1.625	2.560	4.052	6.000
					0.037	0.118	0.209	0.272	0.453	0.807	1.301	2.245	3.333	4.843
N				0.050	0.075	0.125	0.199	0.323	0.495	0.739	1.219	1.920	3.039	4.500
				0.017	0.054	0.067	0.138	0.238	0.362	0.594	1.007	1.454	2.315	3.649
O			0.022	0.033	0.050	0.083	0.133	0.215	0.330	0.492	0.812	1.280	2.026	3.000
			0.007	0.024	0.028	0.059	0.101	0.159	0.265	0.391	0.651	1.058	1.584	2.369
P		0.007	0.013	0.020	0.030	0.050	0.080	0.129	0.198	0.296	0.488	0.768	1.216	1.800
		0.002	0.008	0.011	0.021	0.037	0.069	0.098	0.163	0.236	0.383	0.611	1.044	1.510
Q	0.001	0.003	0.006	0.010	0.015	0.025	0.040	0.065	0.099	0.148	0.244	0.384	0.608	0.900
	0.000	0.002	0.003	0.006	0.013	0.019	0.030	0.053	0.078	0.124	0.205	0.308	0.511	0.790

[1/] The mean square errors are computed for the process average equal to the AQL. If the unbiased estimate is denoted by p_1 and the biased estimate by p_b, the top figure in each cell is $E(p_1\text{-AQL})^2$ and the bottom figure is $E(p_b\text{-AQL})^2$, i.e., the average value of $(p_1\text{-AQL})^2$ and the average value of $(p_b\text{-AQL})^2$ respectively.

Note: The figures for the "mean square errors" are based upon the estimates of the percent defective (rather than fraction defective).

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